where

$$K \equiv \frac{16}{N_{Bo}} \qquad L_T \equiv (1 - \gamma M_0^2)\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2}$$

$$N_{Bo} \equiv \frac{\gamma R \rho_0 U_0}{(\gamma - 1)\sigma T_0^3}$$

which is called the Boltzmann number.

If we proceed in similar fashion from the one-dimensional unsteady-flow equations, it can be shown that the linearized equation with a coordinate system fixed relative to the undisturbed fluid is§

$$\frac{\partial^3 W_s}{\partial x^2 \partial t} + k\alpha_0 \frac{\partial^2 W_T}{\partial x^2} - 3\alpha_0^2 \frac{\partial W_s}{\partial t} = 0 \tag{29}$$

where

$$W_s \equiv \frac{\partial^2\phi}{\partial t^2} - a_0^2 \frac{\partial^2\phi}{\partial x^2} \qquad W_T \equiv \frac{\partial^2\phi}{\partial t^2} - \frac{a_0^2}{\gamma}\frac{\partial^2\phi}{\partial x^2}$$

$$k \equiv \frac{16\gamma a_0}{N_{Bo}}$$

a_0 is the isentropic speed of sound and $N_{Bo} \equiv [\gamma R\rho_0 a_0/(\gamma - 1)\sigma T_0^3]$.

Equations (28) and (29) are the governing differential equations in terms of the velocity potential for two-dimensional steady flows (coordinate system fixed in the body) and one-dimensional unsteady flows (coordinate system fixed in space) with a small departure from radiation equilibrium. It is seen that they are both of fifth order. Unlike the situation in classical equilibrium-flow theory, however, the structure of these two equations is inherently different. This is as would be expected because of the directional properties of radiation intensity. The relationship of the equations to those of classical theory is apparent: In the limits of a very cold gas ($N_{Bo} \to \infty$), a transparent gas ($\alpha_0 \to 0$), or an opaque gas ($\alpha_0 \to \infty$), Eqs. (28) and (29) reduce to the classical equations. For the limiting case of a very hot gas ($N_{Bo} \to 0$), they also reduce to the classical equations but with the isentropic speed of sound replaced by the isothermal speed of sound. The author has been able to apply the linearized equations (29) and (28) to problems of wave propagation and flow over a wavy wall. The results will be presented at a later time.

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§ During the writing of this note, the author came upon a recent paper by Lick.[11] By using the exponential approximation applicable in the one-dimensional case, Lick also obtained Eq. (29) for one-dimensional flow with radiation. This is as would be expected, since the exponential approximation and the moment method applied to one-dimensional problems are mathematically equivalent.

Evaporation Coefficients from Exposure of a Solid to Laser Radiation

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THE theoretical calculation of ablation rates in an intense radiation field is generally a complicated problem, because it involves the interaction of various relaxation phenomena (melting, sublimation or evaporation, ionization), effusive flow or "explosive blowoff" in a complex geometry, and radiative energy transfer. However, under suitable exposure to a laser beam, these processes are effectively decoupled, and a simple phenomenological description is possible for the expected linear regression rate.

Typical relaxation times in the solid phase are of the order of the reciprocal frequency of an equivalent Einstein or Debye oscillator, i.e., of the order of 3×10^{-13} sec. These time estimates indicate that a meaningful temperature can be defined, within a volume element containing very many atoms, on exposure to an intense laser beam with duration longer than about 10^{-9} sec.

For a subliming solid, the linear regression rate (dx/dt) is given by the expression[1]

$$dx/dt = j_e/n^{1/3} \leq (p_s/\rho)(m/2\pi kT)^{1/2} \tag{1}$$

where $j_e \simeq \nu \exp(-\Delta h/kT)$ is the evaporation or sublimation frequency, n is the number of atoms or molecules per unit volume in the solid phase (i.e., $n = \rho/m$), p_s is the saturation vapor pressure at the temperature T of the solid with density ρ, m is the mass per atom, k stands for the Boltzmann constant, ν is the frequency of the Einstein oscillator corresponding to the solid, and Δh represents the heat of evaporation or sublimation per atom or molecule. The upper bound in Eq. (1) corresponds to the limit set by the Knudsen equation with unit evaporation coefficient $\mathcal{K}$. This bound may be made explicit by assuming the validity of Trouton's rule. Thus $p_s \simeq p_0 \exp(-\Delta h/kT)$, where the constant p_0 may be evaluated conveniently by assuming that $p_s = 1$ atm at the normal boiling point $T = T_b$. Hence, $p_s \simeq 2.2 \times 10^{10} \exp(-\Delta h/kT)$ dyne/cm², where we have set $\Delta h/kT_b = 10$, and

$$\nu/n^{1/3} \leq 2.2 \times 10^{10}(1/\rho)(m/2\pi kT)^{1/2} \exp(-\Delta h/kT)$$

Representative estimates of the Knudsen limits (with an evaporation coefficient of unity) for the linear regression rates of metals and stable compounds near the normal boiling points lead to values of a few centimeters per second, i.e., evaporation is unimportant during radiant-energy input for laser pulses with a duration of less than about 10^{-3} sec.

The evaporation coefficient may be defined by the relation

$$\mathcal{K} \equiv \frac{(dx/dt)}{(p_s/\rho)(m/2\pi kT)^{1/2}} \equiv \frac{[\alpha \exp(-\Delta h/kT)]/n^{1/3}}{(p_s/\rho)(m/2\pi kT)^{1/2}} \tag{2}$$

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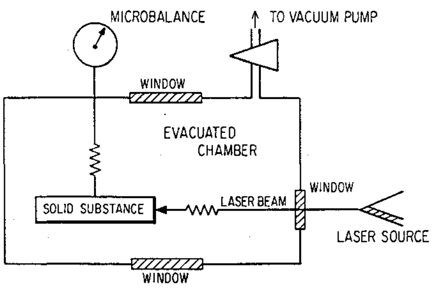

Fig. 1 An apparatus for the experimental determination of evaporation coefficients using an external laser source for heating. The rate of ablation may be measured either by weighing the sample directly or else by observing the surface regression rate through the use of a suitable optical system.

where we have set $j_e = \alpha \exp(-\Delta h/kT)$ and α is an empirically determined frequency at the surface temperature T. In view of the difficulties involved in estimating $\mathcal{H}$ by the use of conventional procedures, it is of obvious interest to use evaporation rate measurements on exposure to a monochromatic laser beam for the experimental measurement of dx/dt and hence of $\mathcal{H}$ under controlled conditions. A schematic diagram illustrating the proposed experimental technique is shown in Fig. 1. The reduction of experimental results for the direct measurement of α is briefly discussed in the following paragraphs.

We assume applicability of a regression law of the form

$$dx/dt \equiv (\alpha/n^{1/3}) \exp[-\Delta h/k(T_0 + \Delta T)] \tag{3}$$

where $T \equiv T_0 + \Delta T$, T_0 denotes the initial temperature, ΔT represents the temperature rise produced at the ablating surface by absorption of radiant energy. The linear absorption coefficient k_l is taken to be a constant independent of temperature. The experiment is performed in such a manner that the temperature at the surface and within the solid does not become large compared with the boiling temperature; the characteristic time for heat conduction t_c is made to be long compared with the test duration t_p. The characteristic time for heat conduction is of the order of $t_c \approx (1/k_l)^2/\alpha_c$ where α_c denotes the thermal diffusivity of the solid. For $t_c \leq t_p$, the coupled problem of energy transfer by conduction and radiation must be considered.[2]

The incident monochromatic radiant energy flux at a distance x below the surface of the absorbing solid is $q_x = q \exp(-k_l x)$. The heat-transfer rate to unit area of the slab of width Δx is $q_{\Delta x} = (-\partial q_x/\partial x)\Delta x$; the heat capacity of a volume element of unit area and thickness Δx is $\rho c_p \Delta x$. Hence, the rate of temperature rise of the slab located at a distance x below the surface becomes

$$(dT/dt)_x = (qk_l/\rho c_p)[\exp(-k_l x)] \tag{4}$$

It is now convenient to introduce the dimensionless variables

$$\left.\begin{aligned} \Delta\theta &= \Delta T/T_0 \\ \tau &= k_l q t/\rho c_p T_0 \\ \eta &= k_l x \\ \theta &= \Delta h/kT_0 \\ B &= \alpha \rho c_p T_0/q n^{1/3} \end{aligned}\right\} \tag{5}$$

Equations (3) and (4) are then reduced to the integrodifferential equation

$$\begin{aligned} d\eta/d\tau &= B \exp-[\theta/(1 + \Delta\theta)] \\ \Delta\theta &= \int_0^{\tau(\eta)} [\exp(-\eta)]d\tau \end{aligned} \tag{6}$$

In order to obtain a solution to the integrodifferential equation by the use of an iteration procedure, we first employ the approximation

$$dx/dt \equiv r_0 = \text{const} \tag{7}$$

in the integrand of Eq. (6), and we find that

$$\left.\begin{aligned} (d\eta/d\tau)_0 &\approx Cr_0 \\ C = \rho c_p T_0/q &= n^{1/3}B/\alpha \\ (\eta)_0 &\approx Cr_0\tau \\ \Delta\theta &\approx \frac{1}{Cr_0}[1 - \exp(-Cr_0\tau)] \end{aligned}\right\} \tag{8}$$

For $Cr_0\tau = k_l r_0 t \ll 1$,

$$d\eta/d\tau \simeq B \exp[\theta/(1 + \tau)]$$

and

$$\begin{aligned} \eta &\simeq B\{(1 + \tau)E_2[\theta/(1 + \tau)] - E_2(\theta)\} \\ E_2(\theta) &= \int_0^1 [\exp(-\theta/\mu)]d\mu \end{aligned} \tag{9}$$

A second iteration to the solution of the problem is obtainable by introducing the integral of Eq. (9) into the integrand appearing in Eq. (6). Actually, this procedure is excessively laborious and it soon becomes preferable to resort to numerical integration.

Since Δh, ρ, c_p, T_0, q, $n^{1/3}$, and k_l are generally easily determined, the experimental determination of x as a function of t implies a measurement of B and hence of α. In laboratory coordinates, Eq. (9) takes for form

$$x = \frac{\alpha \rho c_p T_0}{q k_l n^{1/3}}\left\{\left(1 + \frac{k_l q t}{\rho c_p T_0}\right) E_2\left[\frac{(\Delta h/kT_0)}{1 + (k_l q t/\rho c_p T_0)}\right] - E_2\left(\frac{\Delta h}{kT_0}\right)\right\} \tag{10}$$

for $k_l r_0 t \ll 1$. In view of the large value of $(dT/dt)_x$, it is apparent that t_p must be very small in order to prevent excessive heating and "explosive blowoff." For $qk_l = 4 \times 10^6$ w/cm^3 $= 4 \times 10^{13}$ erg/cm^3-sec, $\rho c_p = 8 \times 10^7$ erg/cm^3 – °K, $\alpha = 10^{12}\,\mathcal{H}'$ sec^{-1}, $n^{1/3} = 2 \times 10^7$ cm^{-1}, and $T_0 = 500$°K, we find that

$$x = 50\mathcal{H}'\{(1 + 10^3 t)E_2[10/(1 + 10^3 t)] - E_2(10)\} \tag{11}$$

where $\mathcal{H}'$ is of the order of magnitude of the evaporation coefficient. With $t = 10^{-3}$ sec and $\mathcal{H}' = 10^{-1}$, the value of x becomes 1×10^{-2} cm and is, therefore, readily observable.

The preceding analysis has been performed on the assumption that the evaporating system remains, at all times, very far from radiative equilibrium. A useful criterion for the existence of radiation equilibrium and for application of a different method of analysis[3] is the statement

$$t_p(dT/dt)_x \geq (\Delta T)_{ss}$$

where $(\Delta T)_{ss}$ is the steady-state temperature rise; for $t_p(dT/dt)_x < (\Delta T)_{ss}$, the numerical estimates derived in the present discussion should be useful. For these specified values, we find from Eq. (4) that $\Delta T \simeq 180$°K for $k_l x = 1$ and $t = 10^{-3}$ sec. Hence, the system remains very far from radiative equilibrium for an intense external laser source, and the temperature rise of a preheated, fairly transparent organic compound is sufficiently small to prevent explosive blowoff. At the same time, the assumed condition $Cr_0\tau \ll 1$ is seen to be satisfied.

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An Improved Glauert Series for Certain Airfoil Problems

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Introduction

In the presence of such complicating effects as unsteadiness, finite span, jet flap, and free-surface interference, for example, an exact solution of the thin airfoil problem is generally out of the question. As a result, it is often convenient to seek a solution for the unknown vortex strength in the form of a truncated Glauert series. In this note we consider the analytical and numerical merits of a slightly modified series solution.

Discussion

Consider, for example, the harmonic motion of an infinitely thin rectangular airfoil of span $2b$ and chord $2a$ in an unbounded inviscid and incompressible stream U. Introducing a right-handed rectangular x', y', z' coordinate system with x' measured downstream, z' normal to the planform, and the origin at the planform centroid, a classical vortex approach leads to the well-known integral equation

$$\bar{w}(x, y) = \frac{1}{4\pi} \int_{-A}^{A} \int_{-1}^{1} \frac{\partial\bar{\gamma}}{\partial\eta}(\xi, \eta) \times \frac{[(x-\xi)^2 + (y-\eta)^2]^{1/2}}{(x-\xi)(y-\eta)} d\xi d\eta + \frac{ike^{ik}}{2\pi} \bar{\Gamma} \int_1^\infty \frac{e^{-ik\xi}}{\xi - x} d\xi + \bar{D}(x, y) \quad (1)$$

where $\gamma(x, y, t) = \bar{\gamma}(x, y) \exp(i\omega t)$ is the unknown strength per unit chordwise length of the spanwise running bound vortices; $\Gamma(y, t) = \bar{\Gamma}(y) \exp(i\omega t)$ is the resulting circulation; $w(x, y, t) = \bar{w}(x, y) \exp(i\omega t)$ is the upwash given over the airfoil; $D(x, y, t) = \bar{D}(x, y) \exp(i\omega t)$ represents a portion of the downwash terms which is regular over the airfoil and, in particular, at the trailing edge; k is the reduced frequency $\omega a/U$; A is the aspect ratio b/a; $\bar{w}$, $\bar{\gamma}$, and D are nondimensionalized with respect to U, $\bar{\Gamma}$ with respect to Ua, and the coordinates with respect to a. The terms on the right represent the downwash which is induced by the vortex system and which must exactly balance $\bar{w}$.

Typically, the equation is then reduced to a more tractable form by suitable approximation of the square root term. For example, the very high (two-dimensional and lifting line[1] theories) and very low[2, 3] aspect ratio theories are equivalent to setting $[(x-\xi)^2 + (y-\eta)^2]^{1/2} \approx |y-\eta|$ and $|x-\xi|$, respectively. Of the various more sophisticated approximations that have appeared in the literature, one of the nicest is due to Laidlaw.[4] He sets

$$[(x-\xi)^2 + (y-\eta)^2]^{1/2} \approx \lambda_0|x-\xi| + \lambda_\infty|y-\eta| \quad (2)$$

where λ_0 and λ_∞ are functions of aspect ratio and are determined by a least-squares minimization of the difference between the left- and right-hand sides of (2) over the planform. Contemplating the use of a Glauert series of the form

$$\bar{\gamma} = c_0(y) \cot\left(\frac{\phi}{2}\right) + \sum_{n=1}^{\infty} c_n(y) \sin n\phi \quad (3)$$

where $x \equiv -\cos\phi$, Laidlaw observes that the right side of (3) vanishes at the trailing edge $\phi = \pi$, whereas the Kutta condition of zero pressure discontinuity demands that $\bar{\gamma} = -ik\bar{\Gamma}$ there. He therefore modifies his Glauert series by adding on $-ik\bar{\Gamma}(y)$. We point out, however, that the latter may be expanded in a Fourier sine series, and, whether we use the modified or unmodified Glauert series, the resulting solutions for $\bar{\gamma}$ will be identical for $\phi < \pi$ although they will equal $-ik\bar{\Gamma}$ and 0, respectively, at $\phi = \pi$. But this discrepancy will not affect the quantities of physical interest so that either series will suffice.

Although the presence of the additional term is immaterial if an exact solution is carried out, it is generally necessary to truncate the summation at $n = N$ and to follow a collocation solution, for example. It is, therefore, of interest to reexamine the potential merits of a more general modified series,

$$\bar{\gamma} = c_0(y) \cot\left(\frac{\phi}{2}\right) + \sum_{n=1}^{N} c_n(y) \sin n\phi + f(\phi, y) \quad (4)$$

in this light. In doing so, let us reformulate the Kutta condition based upon the integral equation; that is, we seek to satisfy the tangent-flow boundary condition at the trailing edge. Observing that, as $x \to 1$, the right side of (1) behaves as

$$\frac{\lambda_\infty}{2\pi} \int_{-1}^{1} \frac{\bar{\gamma}(\xi, y)}{x - \xi} d\xi + \frac{ike^{ik}}{2\pi} \bar{\Gamma} \int_1^\infty \frac{e^{-ik\xi}}{\xi - x} d\xi + 0(1) \quad (5)$$

or

$$-\frac{\lambda_\infty}{2\pi} \bar{\gamma}(1, y) \ln(1 - x) - \frac{ik}{2\pi} \bar{\Gamma} \ln(1 - x) + 0(1) \quad (6)$$

we see that this is only possible if

$$\bar{\gamma}(1, y) = -ik\bar{\Gamma}(y)/\lambda_\infty \quad (7)$$

This integral equation-formulated Kutta condition only coincides with the pressure-formulated condition in the limit of infinite aspect ratio, in which case $\lambda_\infty \to 1$, the incompatibility arising as a consequence of the Laidlaw approximation (2). From (4) and (7) we see that we should choose $f(\pi, y) = -ik\bar{\Gamma}(y)/\lambda_\infty$, but we should not choose $f(\phi, y) = \text{const} = f(\pi, y)$, since this would introduce a $\ln(1 + x)$ singularity from the Cauchy term at the leading edge. To avoid this, we choose $f(0, y) = 0$ so that a suitable choice would be, for example,

$$f(\phi, y) = -ik\bar{\Gamma}(y)\phi/\pi\lambda_\infty \quad (8)$$

We see that Laidlaw's additional term does not remove the logarithmic downwash singularity except in the lifting line or two-dimensional limit and, in fact, introduces one at the leading edge. The properly modified series, (4) and (8), does, on the other hand, lead to a well-defined collocation problem.

In order to assess the numerical importance of the additional term, let us apply the modified and unmodified series to the two-dimensional problem of an airfoil performing vertical translation oscillations since the exact solution is well known[5] and affords a means of comparison. Collocating at $x = 0$, ± 0.4, ± 0.8, we find that, for $k = 0.5$ and 1.0,† the lift predicted using an unmodified series is in error by 12.4 and 19.2%, respectively, compared to 0.3 and 0.1% using the modified series given by (4) and (8) with $\lambda_\infty = 1$.

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† Reduced frequencies in excess of unity are typical in unsteady propeller theory,[6] for example.